Magnetogasdynamic deflagration and detonation waves with ionization

By J. B. HELLIWELL
Department of Mathematics, The Royal College of Science and Technology, Glasgow

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The propagation of a one-dimensional combustion wave into a non-ionized gas at rest in the presence of an electromagnetic field is considered when ionization of the gas occurs across either the combustion wave or a preceding shock wave. The electric and magnetic fields in the undisturbed gas ahead of the waves are mutually perpendicular and orthogonal to the direction of wave propagation. It is shown that steady detonation occurs at a point which is analogous to the Chapman–Jouguet point of ordinary gasdynamic combustion theory. Numerical calculations are made of the state of the gas between and behind the waves in two particlar models, in both of which the upstream electric field is zero. The models are then equivalent to magnetogasdynamic phenomena in a perfectly conducting gas. First, the case of steady detonation is studied. Secondly, steady deflagration in a tube, closed at one end, is discussed.

1. Introduction

The commonly accepted description of a deflagration which has proved adequate for the investigation of steady-state phenomena in non-ionized gases is that of a shock wave which propagates into a cold non-heat conducting, non-viscous gas thereby raising the temperature and pressure so that burning of the hot gas behind it is initiated. In actual fact the burning takes place over a thin section of the gas within which exothermal energy is released, but for theoretical purposes this is replaced by a discontinuity in the gas flow which is termed the combustion wave or flame front. In the case of steady detonation at the Chapman–Jouguet velocity the model is modified so that the shock wave and flame front coalesce into a single discontinuity. For a complete description of the models see, for example, Courant & Friedrichs (1948). Now it is known that ionization of a gas may take place when its temperature attains a value of the order 10^4 deg K. The shock wave in the above models may thus be sufficiently strong to ionize the gas, and if electric or magnetic fields are present the model gives rise to a problem which involves magnetogasdynamic effects, in which interaction occurs between the electromagnetic, calorific and mechanical energies

In this paper it will be supposed that the shock wave is sufficiently strong to fully ionize the gas so that ahead the electrical conductivity $\sigma = 0$, and behind $\sigma = \infty$. In a previous paper Helliwell (1962) has determined some general properties of gas-ionizing shock and combustion waves when the upstream electromagnetic field is parallel to the wave fronts. In this paper more specific results are given for two particular processes. First, it is shown that a steady detonation wave with ionization across the front propagates under the Chapman–Jouguet condition that the velocity of the wave front relative to the burnt fluid particles is equal to the velocity of propagation of small disturbances in the products of combustion. Detailed analysis and calculation are then made for such waves in an initially non-conducting gas in the absence of an electric field. In a sense these calculations extend the result of Gross, Chinitz & Rivlin (1960) who discussed exothermal waves in a gas of infinite conductivity. The second physical process which is analysed is that of a steady deflagration in a tube closed at one end. For simplicity numerical calculations are again made only for the case when the upstream electric field is zero. In such a gas the results are formally identical to those for the magnetogasdynamic deflagration which propagates into a gas which is already fully ionized. The corresponding phenomena in ordinary gasdynamics has been examined by Adams & Pack (1959), among others, and the present studies extend their work into the magnetogasdynamic regime.

2. A model detonation

We consider a one-dimensional model of a detonation wave which propagates with velocity V^* into a non-conducting inviscid gas ($\sigma = 0$) at rest in which the pressure $p = p_0^*$ and the density $\rho = \rho_0^*$. It is supposed that in this gas there is established an electromagnetic field with, relative to absolute axes, an electric field component $\mathbf{E} = \mathbf{E}_0^*$ and magnetic field component $\mathbf{H} = \mathbf{H}_0^*$ both parallel to the wave front and such that $\mathbf{i}$, $\mathbf{E}_0^*$, $\mathbf{H}_0^*$ are mutually orthogonal, where $\mathbf{i}$ is the unit normal vector directed from the wave front into the upstream gas. In what follows we refer to all properties ahead of the wave by the suffix 0, behind the wave by the suffix 2 and measurement in an absolute system of axes by an asterisk. Thus, if we suppose that the temperature rise across the detonation front is sufficient to ionize the gas fully, behind the wave the conductivity $\sigma = \infty$ and the pressure, density, particle velocity, $\mathbf{U}$, and electromagnetic field components are p_2^*, ρ_2^*, $\mathbf{U}_2^*$, $\mathbf{E}_2^*$, and $\mathbf{H}_2^*$. At the detonation front exothermal energy is released by the process of burning but some of this is absorbed by the process of ionization. It is supposed that the overall energy liberated at the front is Q per unit mass. At higher temperatures, corresponding to higher values of Q, further losses of exothermal energy are to be expected due to radiation. No explicit account is taken of such losses in the following analysis.

Now take a set of right-handed orthogonal axes moving with the detonation front such that the x-axis is directed along $\mathbf{i}$. Properties of the gas measured in this set of axes are denoted without asterisk. If we represent by $\mathbf{j}$ and $\mathbf{k}$ the unit vectors along the y- and z-axes, respectively, it follows that

$$\mathbf{E}_0^* = E_0^*\mathbf{j}, \qquad \mathbf{H}_0^* = H_0^*\mathbf{k},$$

$$\mathbf{U}_0 = (U_0^* - V^*)\,\mathbf{i} = -V^*\mathbf{i}, \qquad \mathbf{U}_2 = (U_2^* - V^*)\,\mathbf{i} = -(V^* - U_2^*)\,\mathbf{i},$$

$$\mathbf{E}_0 = (E_0^* - \mu V^* H_0^*)\,\mathbf{j}, \qquad \mathbf{H}_0 = H_0^*\mathbf{k},$$

$$\mathbf{E}_2 = \mathbf{E}_2^* - \mu V^* H_{2z}\mathbf{j} + \mu V^* H_{2y}\mathbf{k}, \qquad \mathbf{H}_2 = \mathbf{H}_2^*,$$

where μ is the permeability, supposed constant, and M.K.S. Giorgi units are used for the measure of electromagnetic quantities. As shown in figure 1, $\mathbf{U}_0$, $\mathbf{U}_2$ are particle velocities and $\mathbf{n}$ is a unit vector, all directed normally from upstream to downstream of the front. The measures of pressure and density are unchanged. Thus

$$p_r = p_r^*, \quad \rho_r = \rho_r^*, \quad r = 0, 2.$$

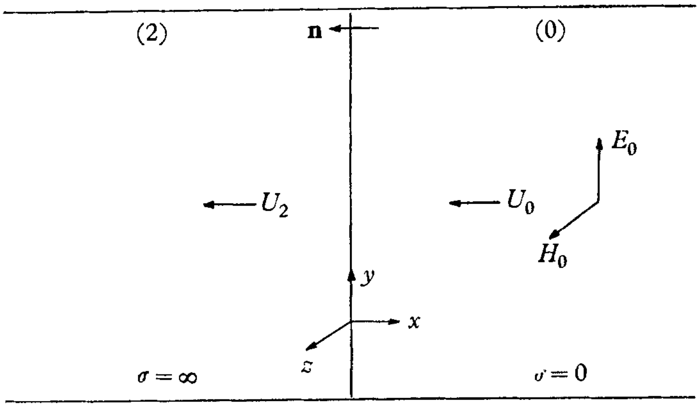

FIGURE 1. Model detonation.

The jump relationships across the front are, in the absence of free magnetic poles, as follows:

$$[H_n] = 0,$$
$$[m] = [\rho U_n] = 0,$$
$$[m\mathbf{U} + (p + \tfrac{1}{2}\mu H^2)\,\mathbf{n} - \mu H_n \mathbf{H}] = 0,$$
$$[m(\tfrac{1}{2}U^2 + p/\rho + \mathscr{E}) + (\mathbf{E}\times\mathbf{H})_n] = mQ,$$
$$[\mathbf{n}\times\mathbf{E}] = 0,$$

where $\mathscr{E}$ is the internal energy per unit mass of the gas due to translation, rotation and vibration of the molecules, and $[X] = X_2 - X_0$. Furthermore, in region 2, if physically unsupportable infinitely large currents are not to arise, we must have, from Ohm's law,

$$\mathbf{E}_2 = -\mu U_2(H_{2z}\mathbf{j} - H_{2y}\mathbf{k}).$$

From these equations it is now straightforward to show that

$$\mathbf{E}_2 = E_2\mathbf{j}, \quad \text{where} \quad E_2 = E_0.$$
$$\mathbf{H}_2 = H_2\mathbf{k}, \quad \text{where} \quad E_2 = -\mu U_2 H_2.$$

The fundamental relationships across the discontinuity may thus we written

$$m = \rho_2 U_2 = \rho_0 U_0, \tag{1a}$$
$$mU_2 + p_2 + \tfrac{1}{2}\mu H_2^2 = mU_0 + p_0 + \tfrac{1}{2}\mu H_0^2, \tag{1b}$$
$$m(\tfrac{1}{2}U_2^2 + p_2/\rho_2 + \mathscr{E}_2) - E_2 H_2 = m(\tfrac{1}{2}U_0^2 + p_0/\rho_0 + \mathscr{E}_0) - E_0 H_0 + mQ, \tag{1c}$$
$$E_0 = E_2 = -\mu U_2 H_2. \tag{1d}$$

Introduce the specific volume $\tau = 1/\rho$, and write

$$p' = p + \tfrac{1}{2}\mu H^2, \quad \mathscr{E}' = \mathscr{E} + \tfrac{1}{2}\mu H^2\tau. \tag{2}$$

Then the jump relationships (1) become

$$m = U_2/\tau_2 = U_0/\tau_0, \tag{3a}$$

$$m(U_2-U_0)+(p_2'-p_0') = 0, \tag{3b}$$

$$\tfrac{1}{2}(U_2^2-U_0^2)+(\mathscr{E}_2'+p_2'\tau_2)-(\mathscr{E}_0'+p_0'\tau_0) = Q', \tag{3c}$$

where $mQ' = mQ - E_0^* H_0^*$. This is the excess of the energy per unit area liberated at the detonation wave over the flux per unit area of the energy of the electromagnetic field ahead of the wave. Now equations (3) are identical with the standard jump relationships across a flame front in ordinary gasdynamics for a fictitious gas in which the pressure p' and internal energy $\mathscr{E}'$ are related by an equation of state of the form $p'\tau - \mathscr{E}' = p\tau - \mathscr{E}$ and exothermal energy is released of amount Q' per unit mass. Provided that this equation of state $\mathscr{E}' = f(p',\tau)$ for the fictitious gas is qualitatively similar to $\mathscr{E} = g(p,\tau)$ appropriate to the real gas, so that the derivatives and curvature have the same sign, then the results of ordinary gasdynamic detonation theory may be taken over into the present theory. It does not seem to be a simple matter to show that this is always so when a discontinuity of electrical conductivity occurs across the front and possibly an analysis of shock structure is required to resolve the question.† This analysis is not carried out here and we proceed on the assumption that the forms are qualitatively similar. The analysis of gasdynamic detonation waves is classical. Reference has already been made to Courant & Friedrichs (1948). For such waves to exist $Q' > 0$. Thus gas-ionizing detonation waves will only exist if the exothermal energy liberated at the front is sufficiently large. The analogy may be taken further and used to derive additional properties. In particular a steady-state detonation is associated with the Chapman–Jouguet point on the combustion adiabatic where $U_2 = C_2$ is the velocity of propagation of small disturbances in the products of combustion. Since in this region $\sigma_2 = \infty$ it follows that $d(H_2\tau_2) = 0$. Hence $dH_2/d\tau_2 = -H_2/\tau_2$. Thus

$$C_2^2 = \frac{dp_2'}{d\rho_2} = -\tau_2^2\left(\frac{dp_2}{d\tau_2}+\mu H_2\frac{dH_2}{d\tau_2}\right) = A_2^2+B_2^2,$$

where A_2, B_2 $\{= (\mu H_2^2\tau_2)^{\frac{1}{2}}\}$ are the speed of sound and Alfvén speed respectively behind the detonation front.

Magnetogasdynamic detonation

In their studies of the structure of a gas-ionizing shock wave Lyubimov & Kulikovsky (1960) and Zhilin (1960) have shown that an electromagnetic wave is propagated into the upstream gas and, in order that the discontinuity may be the limit of some continuous transition, arbitrary values of the electric and magnetic fields cannot be specified. However, they point out that a possible specification is that of zero electric and arbitrary transverse magnetic field ahead of the shock wave. Since the upstream gas is at rest it follows that, in this case, the electric field relative to the particles of gas is zero, which is the usual magnetogasdynamic condition that must hold in a perfectly conducting medium.

† I am indebted to a referee for this remark.

Thus, in such a situation, the effect of ionization will be apparent only in a reduction of the value of the energy release term, Q, due to the use of some of the exothermal energy to ionize the gas. Such detonation waves we then term magnetogasdynamic. Indeed, since $E_0^* = 0$ we have

$$E_0 = -\mu U_0 H_0. \tag{4}$$

If we put $\tau = 1/\rho$ and eliminate E_0, E_2 between the equations (1) and (4) we obtain the appropriate jump relationships in the form

$$m = U_2/\tau_2 = U_0/\tau_0, \tag{5a}$$

$$m(U_2 - U_0) + (p_2 + \tfrac{1}{2}\mu H_2^2) - (p_0 + \tfrac{1}{2}\mu H_0^2) = 0, \tag{5b}$$

$$\tfrac{1}{2}(U_2^2 - U_0^2) + (\mathscr{E}_2 + p_2\tau_2 + \mu H_2^2\tau_2) - (\mathscr{E}_0 + p_0\tau_0 + \mu H_0^2\tau_0) = Q. \tag{5c}$$

The additional condition for detonation at the Chapman–Jouguet point is

$$U_2^2 = C_2^2 = A_2^2 + \mu H_2^2\tau_2. \tag{6}$$

Magnetogasdynamic detonation in a perfect gas

As a final simplification of the model it is supposed that the gas on either side of the detonation front is perfect but that the adiabatic index, γ, takes different constant values γ_0, γ_2 in the upstream and downstream regions. Then the internal energy, $\mathscr{E}_r = p_r\tau_r/(\gamma_r - 1)$ for $r = 0, 2$. In the region upstream of the wave the speed is denoted by $a_0 = (\gamma_0 p_0 \tau_0)^{\frac{1}{2}}$. The magnitudes of the particle, wave, sound and Alfvén velocities are now written in terms of a_0 as unit of measure. Thus upstream of the front we introduce a dimensionless Alfvén speed, α, given by $\alpha^2 = \mu H_0^2\tau_0/a_0^2$. The additional non-dimensional speeds v^*, c_2, a_2 are defined as follows:

$$U_0 = V^* = v^* a_0, \quad U_2 = C_2 = c_2 a_0, \quad A_2 = a_2 a_0. \tag{7}$$

Then the jump relationships (5), which give the flow and electromagnetic parameters behind the wave in terms of those ahead, become

$$c_2/v^* = \tau_2/\tau_0, \tag{8a}$$

$$\gamma_0 v^*(c_2 - v^*) + (\gamma_0\tau_0/\gamma_2\tau_2)\,a_2^2 - 1 + \tfrac{1}{2}\gamma_0\alpha^2\{(\tau_0/\tau_2)^2 - 1\} = 0, \tag{8b}$$

$$(c_2^2 - v^{*2}) + 2a_2^2/(\gamma_2 - 1) - 2/(\gamma_0 - 1) + 2\alpha^2\{(\tau_0/\tau_2) - 1\} = 2Q/a_0^2, \tag{8c}$$

$$c_2^2 = a_2^2 + \alpha^2(\tau_0/\tau_2), \tag{8d}$$

together with

$$H_2^* = H_0^*(\tau_0/\tau_2), \quad E_2^* = v^*(\tau_0/\tau_2 - 1), \tag{8e}$$

where H_0^* is arbitrary and E_0^* is zero.

We note that, in the case when $\gamma_2 = \gamma_0$ equations ($8\,a, b, c, d$) with considerable changes of notation are essentially those discussed by Gross *et al.* (1960) as leading to the 'singular solution at the Chapman–Jouguet point' of their paper. In the present paper we give an independent and distinct analysis of these equations from which numerical details of the properties of steady magnetogasdynamic detonation waves are drawn for several values of Q/a_0^2 and an associated wide range of values of α. These considerably amplify the results of the earlier work. In the following section we have occasion to refer to these numerical details for comparative purposes.

From equations (8) we eliminate a_2 and c_2. Thus

$$\left\{\frac{\gamma_0}{\gamma_2}(\gamma_2+1)-\gamma_0\frac{\tau_0}{\tau_2}\right\}v^{*2}=\frac{\tau_0}{\tau_2}\left\{\frac{\gamma_0}{2\gamma_2}(2-\gamma_2)\alpha^2\left(\frac{\tau_0}{\tau_2}\right)^2+1+\frac{\gamma_0}{2}\alpha^2\right\}, \tag{9a}$$

$$\left\{\frac{\gamma_2+1}{\gamma_2-1}-\left(\frac{\tau_0}{\tau_2}\right)^2\right\}v^{*2}=\left(\frac{\tau_0}{\tau_2}\right)^2\left\{2\alpha^2\left(\frac{2-\gamma_2}{\gamma_2-1}\right)\frac{\tau_0}{\tau_2}+2\alpha^2+\frac{2}{\gamma_0-1}+\frac{2Q}{a_0^2}\right\}. \tag{9b}$$

The elimination of v^* then leads to the fourth-degree equation for the density ratio (τ_0/τ_2) as follows:

$$(\tau_0/\tau_2)^4+h_1(\tau_0/\tau_2)^3+h_2(\tau_0/\tau_2)^2+h_3(\tau_0/\tau_2)+h_4=0, \tag{10}$$

where

$$h_1=-4\gamma_2/(\gamma_2-1),$$

$$h_2=-2[\{\gamma_2(\gamma_0+1)\}/\{(\gamma_0(\gamma_0-1)\}+2\gamma_2Q/a_0^2+3\alpha^2(\gamma_2^2-\gamma_2-1)/(\gamma_2-1)]/[(2-\gamma_2)\alpha^2],$$

$$h_3=4(\gamma_2+1)[1/(\gamma_0-1)+Q/a_0^2+\alpha^2]/[(2-\gamma_2)\alpha^2],$$

$$h_4=-[2\gamma_2(\gamma_2+1)(1+\tfrac{1}{2}\gamma_0\alpha^2)]/[\gamma_0(\gamma_2-1)(2-\gamma_2)\alpha^2].$$

It remains to distinguish which of the four roots of this quartic is that relevant to the solution of the detonation problems.

We recall that $1<\gamma_0,\gamma_2<2$. Now for a physically realistic solution v^* itself must be real, that is $v^{*2}\geqslant 0$. Hence from equation (9*a*), since the right-hand side is always positive, $\tau_0/\tau_2\leqslant(\gamma_2+1)/\gamma_2$. Also from equation (9*b*) we deduce similarly that $\tau_0/\tau_2\leqslant\{(\gamma_2+1)/(\gamma_2-1)\}^{\frac{1}{2}}$. Therefore, since

$$(\gamma_2+1)/\gamma_2<\{(\gamma_2+1)/(\gamma_2-1)\}^{\frac{1}{2}},$$

it follows that across any steady magnetogasdynamic detonation wave the density ratio is restricted to the range

$$0\leqslant\tau_0/\tau_2\leqslant(\gamma_2+1)/\gamma_2, \tag{11}$$

which is independent of α^2 and thus the same as in ordinary gasdynamic detonation theory. Let us now consider the roots of equation (10). First, we notice that for all (Q,α) the coefficients $h_1<0$, $h_3>0$, $h_4<0$. Therefore there is always at least one negative root and an odd number of positive roots. From a consideration of the form of the curves (9*a*, *b*) in the $(v^{*2},\tau_0/\tau_2)$-plane it is apparent that the positive root which always exists lies in the unrealistic range

$$\tau_0/\tau_2>\{(\gamma_2+1)/(\gamma_2-1)\}^{\frac{1}{2}}.$$

Further, in the range

$$0\leqslant\tau_0/\tau_2\leqslant(\gamma_2+1)/\gamma_2<\{(\gamma_2+1)/(\gamma_2-1)\}^{\frac{1}{2}},$$

the curve (9*a*) is concave upwards with positive slope at $\tau_0/\tau_2=0$ and asymptotic to $\tau_0/\tau_2=(\gamma_2+1)/\gamma_2$. Similarly in the range

$$0\leqslant\tau_0/\tau_2\leqslant\{(\gamma_2+1)/(\gamma_2-1)\}^{\frac{1}{2}}$$

the curve (9*b*) is also concave upwards, has zero slope at $\tau_0/\tau_2=0$ and is asymptotic to $\tau_0/\tau_2=\{(\gamma_2+1)/(\gamma_2-1)\}^{\frac{1}{2}}$. But in the case of ordinary gasdynamic flame theory it is known from exact analysis that at the Chapman–Jouguet condition two velocities of propagation occur, namely those corresponding to steady detonation and deflagration, respectively. Thus when $\alpha^2 = 0$ the two curves always intersect in the range $0 < \tau_0/\tau_2 < (\gamma_2+1)/\gamma_2$. However, the rate of increase with α^2 of the slope of the curve (9*b*) is greater than the corresponding rate of increase of the slope of curve (9*a*) in this range. Therefore, for $\alpha^2 \geqslant 0$, two intersections always occur in this range, and since the steady detonation velocity is, by definition, very much greater than the associated steady deflagration velocity it follows that $(\tau_0/\tau_2)_{\text{detonation}} > (\tau_0/\tau_2)_{\text{deflagration}}$. Hence the quartic equation (10) has four real roots and the middle positive root is that relevant to a steady detonation.

The complete solution to a detonation problem is thus obtained by finding the second largest positive root of equation (10) for τ_0/τ_2, determining v^* from either of equations (9*a*, *b*), then c_2 from equation (8*a*), a_2 from equation (8*d*) and E_2^* together with H_2^*/H_0^* from equation (8*e*). Finally the pressure ratio p_2/p_0 follows from $p_2/p_0 = a_2^2(\tau_0/\tau_2)$ and the temperature ratio is given by

$$T_2/T_0 = (c_{V_0}/c_{V_2})\{(\gamma_0-1)/(\gamma_2-1)\}(p_2/p_0)/(\tau_0/\tau_2),$$

where c_{V_0}, c_{V_2} are the specific heats at constant volume on the two sides of the wave front.

For simplicity in the numerical calculations it is assumed that the ratios of the specific heats of the unburnt gas and products of combustion are the same. The value used is $\gamma_0 = \gamma_2 = \frac{5}{3}$ appropriate to a monatomic gas. A range of values of the two parameters $(\alpha^2, Q/a_0^2)$ are taken as follows:

α^2	0	10	50	100	500	1000
Q/a_0^2	50	—	100	—	500	—

These values of α^2 correspond to a range of magnetic field strength of the order $0 \leqslant H_0^* < 10{,}000\,\text{G}$ in a gas at atmospheric pressure ($p_0 = 1\,\text{atm}$) and higher values at higher pressures. With a magnetic field strength of the order of 10,000 G it is possible that the effects of Hall currents may be significant. In order not to confuse the model unduly any such effects have been neglected in this paper. At the lower end of the scale for Q/a_0^2 the value $Q/a_0^2 = 50$ is that for a conventional explosive releasing about 1700 cal/g in a gas at atmospheric pressure and density of 1 g/l. At the upper end of the scale $Q/a_0^2 = 500$ is a value corresponding to thermonuclear fusion in a more diffuse gas. It will thus be noted that the numerical details of the computed models cover an extensive range of physical situations and it should be remarked that certain aspects of these may not be particularly realistic. For instance, in the case of a conventional explosive, dissociation effects may limit the temperature and full ionization behind the front is not likely to occur. Further, except at low degrees of ionization the absorption of ionization energy from the exothermal energy is not necessarily negligible. However, the neglect of this absorption, and the supposition that $\sigma = \infty$ behind the front, leads to a tractable mathematical analysis and calculations based upon a value of $\gamma = \frac{5}{3}$ appropriate to a fully ionized monatomic gas even if not of quantitative interest should give qualitative information to form a basis for more realistic computations in later work. In actual fact the numerical results are not markedly dependent upon the actual choice of the value for γ. The results of the calculations are given in figures 2 and 3. In figure 2 are shown contours at constant Q/a_0^2 of detonation speed and the speed of propagation of small disturbances in the burnt gas, both in terms of the speed of sound in the unburnt gas as unit of measure. In figure 3 are displayed similar contours of the pressure and density ratios together with the magnetic and electric field strengths in the burnt gas referred to a system of absolute axes in which the

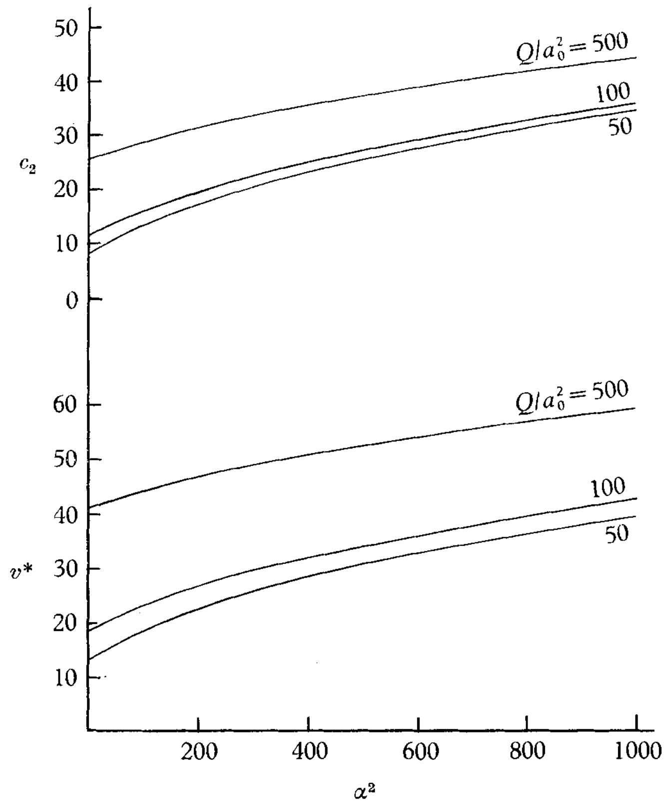

FIGURE 2. Detonation wave: propagation speeds.

unburnt gas is at rest. Contours of temperature have not been shown. These are easily obtained from $T_2/T_0 = (p_2/p_0)/(\rho_2/\rho_0)$. Clearly even if we suppose that the unburnt gas is cold the temperature in the burnt gas attains a value $T > 10^4$ degK which shows that effects of ionization may be significant.

3. A model deflagration

The one-dimensional model of a deflagration which we shall consider is that of a shock wave followed by a combustion wave at which exothermal energy is released. The gas at rest upstream of the shock wave is supposed non-conducting electrically but is fully ionized by the passage of the shock wave. In the upstream region electric and magnetic fields may exist with mutually orthogonal components parallel to the wave fronts. Figure 4 shows a schematic diagram of the model. Suffices 0, 1, 2 refer to the various regions ahead of, between and behind the wave, respectively, U^* is the gas velocity, E_0^*, H_0^* are the magnitudes of the upstream electric and magnetic fields in the directions of the y, z-axes, respectively,

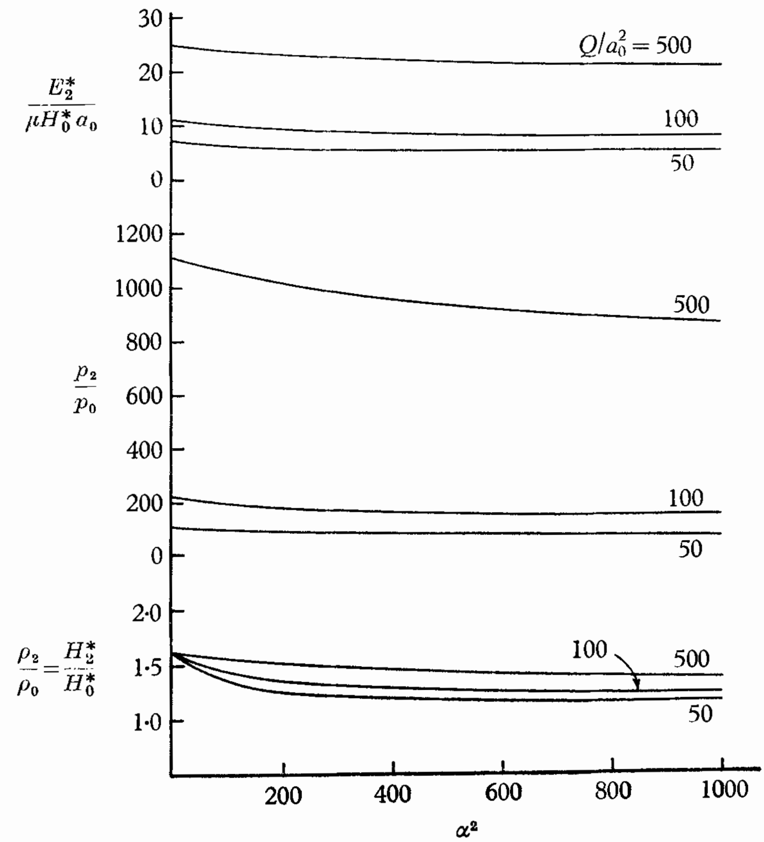

FIGURE 3. Detonation wave: pressure, density and electric field.

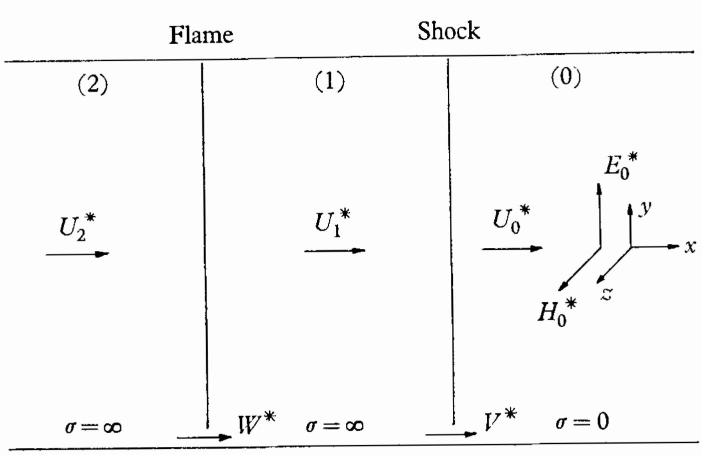

FIGURE 4. Model deflagration.

and V^*, W^* are the velocities of the gas-ionizing shock wave and magnetogasdynamic combustion wave, respectively, all quantities being measured in a system of axes in which the upstream gas is at rest.

The general analysis of this model is similar to that of the previous section. It has been carried through in an earlier paper by Helliwell (1962) in which the various conservation relations across the two waves were laid down and the Hugoniot curves in the (p, τ)-plane established. In this paper the jump relations across the shock wave were presented as equations (4), (5) and across the combustion wave they were given by equations (22). For ease of reference these equations, with minor rearrangement are repeated here. They are as follows:

for the gas-ionizing shock wave

$$m = U_1/\tau_1 = U_0/\tau_0, \tag{12a}$$

$$mU_1 + p_1 + \tfrac{1}{2}\mu H_1^2 = mU_0 + p_0 + \tfrac{1}{2}\mu H_0^2, \tag{12b}$$

$$m(\tfrac{1}{2}U_1^2 + p_1\tau_1 + \mathscr{E}_1) - E_1 H_1 = m(\tfrac{1}{2}U_0^2 + p_0\tau_0 + \mathscr{E}_0) - E_0 H_0, \tag{12c}$$

$$-\mu U_1 H_1 = E_1 = E_0, \tag{12d}$$

together with

$$U_r^* = V^* - U_r, \quad E_r^* = E_r + \mu V^* H_r, \quad H_r^* = H_r, \quad p_r^* = p_r, \quad \rho_r^* = \rho_r, \quad \text{for} \quad r = 0, 1; \tag{13}$$

for the magnetogasdynamic combustion wave

$$m = U_2/\tau_2 = U_1/\tau_1, \tag{14a}$$

$$mU_2 + p_2 + \tfrac{1}{2}\mu H_2^2 = mU_1 + p_1 + \tfrac{1}{2}\mu H_1^2, \tag{14b}$$

$$m(\tfrac{1}{2}U_2^2 + p_2\tau_2 + \mathscr{E}_2) - E_2 H_2 = m(\tfrac{1}{2}U_1^2 + p_1\tau_1 + \mathscr{E}_1) - E_1 H_1 + mQ, \tag{14c}$$

$$-\mu U_2 H_2 = E_2 = E_1 = -\mu U_1 H_1, \tag{14d}$$

together with

$$U_s^* = W^* - U_s, \quad E_s^* = E_s + \mu W^* H_s, \quad H_s^* = H_s, \quad p_s^* = p_s, \quad \rho_s^* = \rho_s, \quad \text{for} \quad s = 1, 2. \tag{15}$$

In these equations all quantities without asterisk refer to axes moving with the shock or combustion wave respectively. (The definitions of U_1, E_1 are thus different for the two waves; there should however be no confusion.) The permeability is supposed constant throughout. The absorption of ionization energy at the shock wave is supposed negligible and the exothermal energy released at the combustion front is written Q per unit mass. In the case of a high degree of ionization the absorption of ionization energy at the shock wave is unlikely to be negligible compared with the exothermal energy released at the combustion front. However, in this paper, the jump of conductivity from $\sigma = 0$ to $\sigma = \infty$ across the shock wave should be regarded as a scale effect rather than a consequence of full ionization of the gas, in which case the neglect of the absorption of ionization energy is not unreasonable. Its insertion would introduce an additional parameter into an already complicated, yet highly idealized, model for which any numerical results should in any case be regarded as qualitative rather than quantitative. The electric field, $\mathbf{E}$, and magnetic field, $\mathbf{H}$, do not rotate from their original directions in the upstream gas.

Magnetogasdynamic deflagration in a perfect gas

In the present study, for simplicity of algebraic manipulation and succeeding computation, we restrict consideration to the case when the upstream electric field is absent. Then, as remarked in the previous section, the equations and analysis become formally identical to the case of a model deflagration of the above type which propagates into a gas already perfectly conducting. Thus in equations (13) we set $E_0^* = 0$. Further, it is supposed that the static gas into which the deflagration propagates is perfect with constant ratio of specific heats, γ_0, and that the gas remains perfect throughout the entire field but that the effect of the ionization and combustion is to change the value of γ_0 to γ_1, γ_2 in the appropriate regions. The internal energy of the gas in the various regions is given by $\mathscr{E}_r = p_r\tau_r/(\gamma_r - 1)$ with $r = 0, 1, 2$ and in regions 1, 2 the speed of sound, A_r, is given by $A_r^2 = \gamma_r p_r \tau_r$ for $r = 1, 2$. The speed of sound in the upstream region 0 is denoted by a_0, where $a_0^2 = \gamma_0 p_0 \tau_0$ and is used as unit of measure for all velocities. Thus the dimensionless Alfvén speed, α, upstream of the waves is defined by

$$\alpha^2 = \mu H_0^2 \tau_0 / a_0^2. \tag{16}$$

The additional non-dimensional speeds v^*, w^*, a_1, a_2, u_r, u_r^* are introduced by the relationships

$$V^* = v^* a_0, \quad W^* = w^* a_0, \quad A_1 = a_1 a_0, \quad A_2 = a_2 a_0, \quad U_r = u_r a_0, \quad U_r^* = u_r^* a_0, \tag{17}$$

for $r = 0, 1, 2$. After a little algebra the jump relationships across the two waves may thus be described in the following forms.

For the gas-ionizing shock wave:

$$\frac{v^* - u_1^*}{v^*} = \frac{\tau_1}{\tau_0} = \frac{H_0^*}{H_1^*}, \tag{18a}$$

$$v^* u_1^* = \left(\frac{a_1^2 \tau_0}{\gamma_1 \tau_1} - \frac{1}{\gamma_0}\right) + \frac{\alpha^2}{2}\left\{\left(\frac{\tau_0}{\tau_1}\right)^2 - 1\right\}, \tag{18b}$$

$$v^{*2} - (v^* - u_1^*)^2 = 2\left(\frac{a_1^2}{\gamma_1 - 1} - \frac{1}{\gamma_0 - 1}\right) + 2\alpha^2\left(\frac{\tau_0}{\tau_1} - 1\right). \tag{18c}$$

The strengths of the electric fields are given by

$$E_0^* = 0, \quad E_1^*/(\mu H_0^* a_0) = v^*(\tau_0/\tau_1 - 1). \tag{18d}$$

The pressure ratio is

$$\frac{p_1}{p_0} = \frac{\gamma_0}{\gamma_1}\left(\frac{\tau_0}{\tau_1}\right) a_1^2. \tag{18e}$$

For the magnetogasdynamic combustion wave

$$\frac{w^* - u_2^*}{w^* - u_1^*} = \frac{\tau_2}{\tau_1} = \frac{H_1^*}{H_2^*}, \tag{19a}$$

$$(w - u_2^*)(u_1^* - u_2^*) = \frac{a_1^2}{\gamma_1}\left(\frac{\tau_2}{\tau_1}\right) - \frac{a_2^2}{\gamma_2} + \frac{\alpha^2}{2}\left(\frac{\tau_0}{\tau_1}\right)\left\{\frac{\tau_2}{\tau_1} - \frac{\tau_1}{\tau_2}\right\}, \tag{19b}$$

$$(w^* - u_1^*)^2 - (w^* - u_2^*)^2 = 2\left(\frac{a_2^2}{\gamma_2 - 1} - \frac{a_1^2}{\gamma_1 - 1}\right) + 2\alpha^2\left(\frac{\tau_0}{\tau_1}\right)\left(\frac{\tau_1}{\tau_2} - 1\right) - \frac{2Q}{a_0^2}. \tag{19c}$$

The electric fields are given by

$$\frac{E_1^*}{\mu H_0^* a_0} = v^*\left(\frac{\tau_0}{\tau_1}-1\right), \quad \frac{E_2^*-E_1^*}{\mu H_0^* a_0} = w^*\frac{\tau_0}{\tau_1}\left(\frac{\tau_1}{\tau_2}-1\right), \tag{19d}$$

and the pressure ratio is

$$\frac{p_2}{p_1} = \frac{\gamma_1}{\gamma_2}\left(\frac{\tau_1}{\tau_2}\right)\left(\frac{a_2}{a_1}\right)^2. \tag{19e}$$

For a given upstream state of the unburnt, non-ionized gas, equations (18) are a set of six equations for the seven variables v^*, u_1^*, p_1, τ_1, a_1, H_1^*, E_1^* which define the conditions between the shock and combustion wave, and for a known rate of exothermal energy release, Q/a_0^2, equations (19) are a similar set of six equations for the seven variables w^*, u_2^*, p_2, τ_2, a_2, H_2^*, E_2^* in the products of combustion. The solution of the problem thus has two degrees of freedom; if any two of the quoted variables are specified the problem has a unique solution. For instance, for given speeds of the combustion and shock waves the flow parameters and electromagnetic field are completely determined. In the subsequent development we shall suppose that an alternative pair of conditions are specified, viz. the speed v^* of the gas-ionizing shock wave and the speed u_2^* in the burnt gas. In the final calculations we shall set $u_2^* = 0$ corresponding to deflagration in a closed tube, but meantime as a more general investigation we retain u_2^* in the analysis. However, to avoid undue algebraic complication in what follows it is supposed that the adiabatic index is constant through the entire flow. Thus the suffix is removed on γ_r for all r, and we write

$$\gamma_0 = \gamma_1 = \gamma_2 = \gamma.$$

Consider first the gas-ionizing shock wave described by equations (18) in which it is supposed that v^* is specified. Then, by simple elimination, it can be shown that the particle velocity u_1^* is given by the solution of the quadratic equation

$$(\gamma+1)\,v^*u_1^{*2}-\{(\gamma+3)\,v^{*2}-2-\gamma\alpha^2\}\,u_1^*-2v^*\{1-v^{*2}-\alpha^2\} = 0.$$

provided that $u_1^* \neq 0$. For general v^*, the roots of this equation are

$$2(\gamma+1)\,v^*u_1^* = [(\gamma+3)\,v^{*2}-2-\gamma\alpha^2] \pm \{[(\gamma+3)\,v^{*2}-2-\gamma\alpha^2]^2-8(\gamma+1)\,v^{*2}(v^{*2}-1-\alpha^2)\}^{\frac{1}{2}}, \tag{20}$$

which may be written in the form

$$2(\gamma+1)\,v^*u_1^* = [(\gamma+3)\,v^{*2}-2-\gamma\alpha^2] \pm \{[(\gamma-1)\,v^{*2}+2+\gamma\alpha^2]^2-4\alpha^2v^{*2}(\gamma-2)\,(\gamma+1)\}^{\frac{1}{2}}.$$

Thus for values of $\gamma < 2$ the positive sign yields $u_1^* \geqslant v^*$ according as $\alpha^2 \geqslant 0$, that is, the velocity of the gas behind the shock wave is greater than or equal to the velocity of the wave itself, which is clearly impossible. Furthermore, if the negative sign is taken, the corresponding value of $u_1^* < v^*$ for all α^2. Finally, since for a real wave $u_1^* > 0$ the velocity of the shock wave has a lower bound given by $v^* > (1+\alpha^2)^{\frac{1}{2}}$. In the case $v^* = (1+\alpha^2)^{\frac{1}{2}}$ the shock wave degenerates into an acoustic wave with the gas particles at rest behind it. Once the value of u_1^* has been determined from equation (20), taking the negative sign, the associated density and magnetic field behind the shock wave are found from equation (18a). The electric field is then obtained from equation (18d). The elimination of (τ_1/τ_0) from equations (18a, b) leads to the expression

$$a_1^2 = \frac{v^* - u_1^*}{v^*}\left\{\gamma v^* u_1^* + 1 - \frac{\gamma\alpha^2}{2}\left[\left(\frac{v^*}{v^* - u_1^*}\right)^2 - 1\right]\right\}, \tag{21}$$

for the speed of sound behind the shock, from which, by use of equation (18e) the pressure is evaluated.

In a similar manner we may examine the magnetogasdynamic combustion wave the properties of which are stated as equations (19), where all variables bearing suffix 1 are supposed known from the above solution for the shock wave, and the downstream particle velocity, u_2^*, is specified. In this case the algebra is similar to the above but more formidable. The elimination of (τ_2/τ_1) and a_2 from equations (19a, b, c) leads to the following cubic equation for the speed, w^*, of the combustion wave,

$$(w^* - u_1^*)(w^* - u_2^*)\left[w^* + \frac{\gamma - 1}{2}u_1^* - \frac{\gamma+1}{2}u_2^* + \frac{(\gamma-1)Q}{a_0^2(u_1^* - u_2^*)}\right]$$
$$- a_1^2(w^* - u_2^*) - \alpha^2\left(\frac{\tau_0}{\tau_1}\right)\left[w^* + \frac{\gamma-2}{2}u_1^* - \frac{\gamma}{2}u_2^*\right] = 0. \tag{22}$$

Now when $\alpha^2 = 0$, the equation degenerates into the gasdynamic case and one root is $w^* = u_2^*$; of the others one is negative and the remaining positive root corresponds to the combustion velocity. Further, if $u_2^* \geqslant 0$, this cubic equation has two positive and one negative root, since for a real combustion wave $u_1^* > u_2^*$ and $\gamma < 2$. Therefore since the solution is a continuously varying function of α^2 we deduce that the speed, w^*, of the combustion wave, $u_2^* < w^* < u_1^*$, is given by the largest positive root of equation (22). In the region of burnt gas the density and magnetic field are then given by equations (19a) and the electric field is obtained from equation (19d). The speed of sound, a_2, is then derived from equation (19b) and finally the pressure from equation (19e).

In conclusion, it is of interest to calculate the speed of propagation of small disturbances, C_1, C_2, in the region between the waves and behind the combustion wave, respectively. These are given by

$$C_r^2 = A_r^2 + \mu H_r^2 \tau_r \quad (r = 1, 2).$$

Thus in non-dimensional form, writing $C_r = c_r a_0$, we have

$$c_1^2 = a_1^2 + \alpha^2(\tau_0/\tau_1), \quad c_2^2 = a_2^2 + \alpha^2(\tau_0/\tau_2). \tag{23}$$

The importance of these quantities lies in the fact seen earlier that when the velocity of the combustion front relative to the products of combustion has the value C_2 detonation may occur under Chapman–Jouguet conditions. Thus, by analogy with the concepts of ordinary gasdynamic detonation theory, deflagration may be only expected to occur when $w^* - u_2^* \leqslant c_2$, and the mathematically feasible 'strong deflagration' solutions of the above when $w^* - u_2^* > c_2$ are believed to have no physical reality.

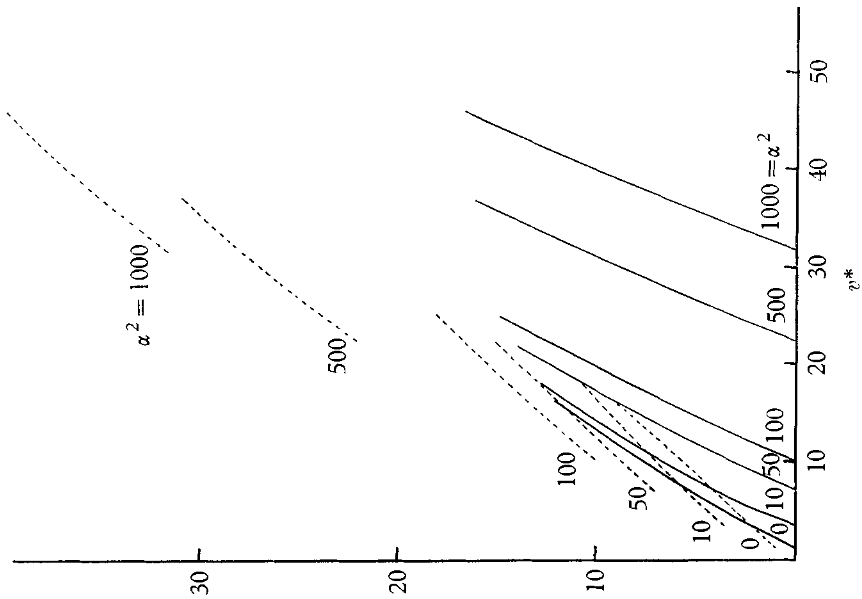

FIGURE 6. Unburnt ionized gas speeds. ——, u_1^*; - - - -, c_1.

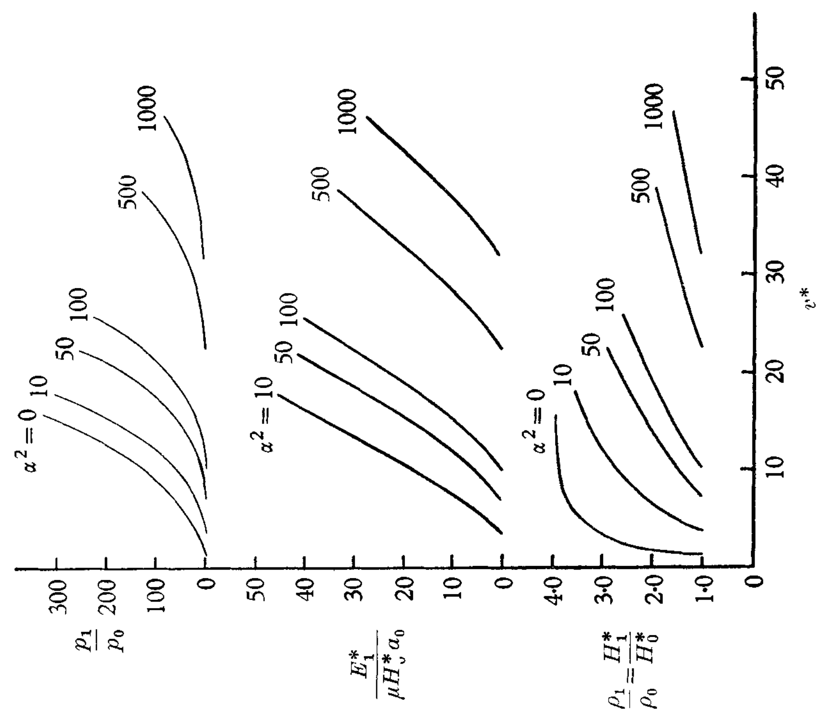

FIGURE 5. Unburnt ionized gas state.

Magnetogasdynamic deflagration in a closed tube

In a previous paper Adams & Pack (1959) investigated the deflagration of a gas in a closed tube in a purely gasdynamic context. As an illustration of the theory developed in this paper a detailed numerical examination is now made of deflagration in a closed tube in the presence of a magnetic field with ionization across the shock wave. The results of Adams & Pack are included as a special case. In view of the remarks at the end of the preceding paragraph, for sufficiently low speeds of deflagration, the model illustrated in figure 4 and subsequently analysed is adequate for this purpose if in it we set $U_2^* = 0$. Since the products of combustion are at rest in the absolute system of axes, $E_2^* = 0$. Explicit analytical solution of the appropriate equations is not possible and recourse must be had to a graphical presentation. The evaluation of the numerous details was carried out on a digital computer for the range of values of α^2 and Q/a_0^2 listed in § 2. The results are shown in the sequence of figures 5 to 12.

Consider first the change of state across the gas-ionizing shock wave. This is illustrated by figures 5, 6 for different strengths of the upstream magnetic field, H_0^*, which we recall is proportional to α. As has already been noted, the minimum speed of the shock wave is given by $v^* = (1+\alpha^2)^{\frac{1}{2}}$ and then the shock is of zero strength. With increasing H_0^*,

(i) both the strength $(p_1-p_0)/p_0$ and the condensation $(\rho_1-\rho_0)/\rho_0$ increase more slowly with increasing shock velocity;

(ii) the temperature ratio, $T_1/T_0 = (p_1/p_0)/(\rho_1/\rho_0)$ rises more quickly with increasing shock velocity.

With fixed H_0^*

(i) the magnitude of the electric field behind the shock is very approximately proportional to the excess shock speed over the minimum;

(ii) for increasing values of the shock speed the particle speed behind the shock as in ordinary gasdynamics, is at first 'subsonic' but for a sufficiently large shock speed becomes 'supersonic' (the term 'sonic' refers to the speed of propagation of small disturbances in the gas behind the shock, which in the presence of a magnetic field is greater than the speed of sound); with increasing H_0^* the transition takes place at increasingly greater values of the shock speed; if H_0^* is large the particle speed is 'subsonic' except possibly for exceedingly fast shocks.

Now consider the overall deflagration. The details of the combustion wave are given in figures 7 to 12. These should be studied in conjunction with figure 2 which shows the properties of a steady detonation. The speed of the combustion wave is always less than that of the preceding shock wave even when the former is moving faster than the speed of the corresponding detonation wave. Thus the feature of ordinary gasdynamic deflagration first noted by Adams & Pack (1959) that attainment of the Chapman–Jouguet velocity by the combustion wave does not lead to detonation is unchanged by magnetogasdynamic effects. The ranges of v^* when $w^* > c_2$ are not discussed—they correspond to 'strong deflagrations' which as remarked earlier are believed to have no physical import. Almost all other features of ordinary gasdynamic deflagration pass unchanged to the magnetogasdynamic case. The speed of the combustion wave increases more

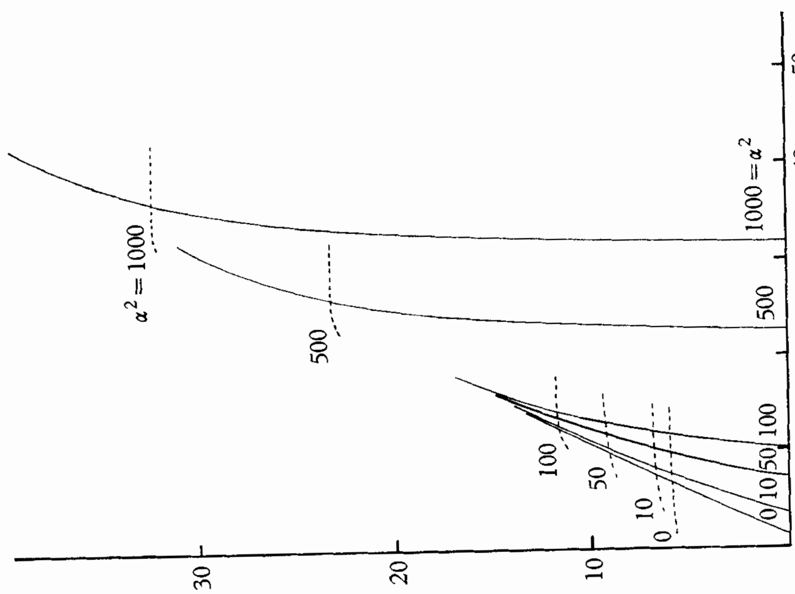

FIGURE 7. Combustion wave speeds ($Q/a_0^2 = 50$).
——, w^*; – – – –, c_2.

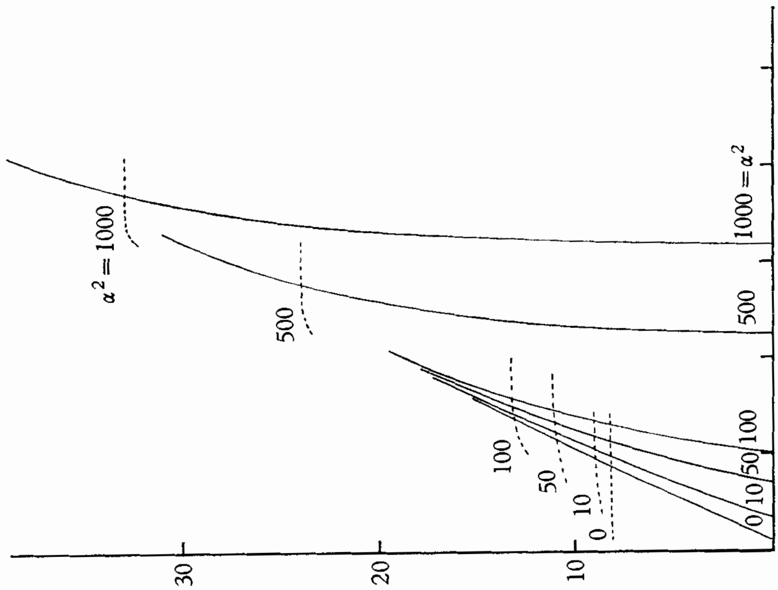

FIGURE 8. Combustion wave speeds ($Q/a_0^2 = 100$).
——, w^*; – – – –, c_2.

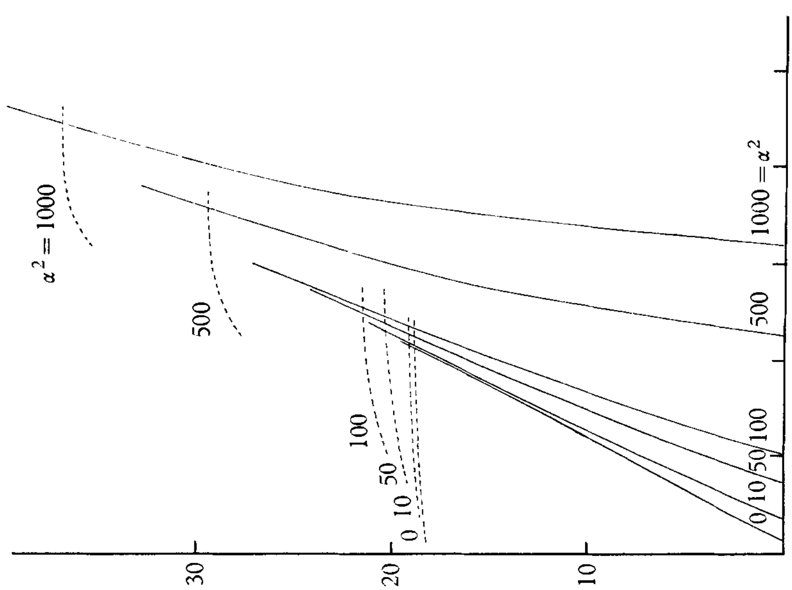

FIGURE 9. Combustion wave speeds ($Q/a_0^2 = 500$).
——, w^*; - - - -, c_2.

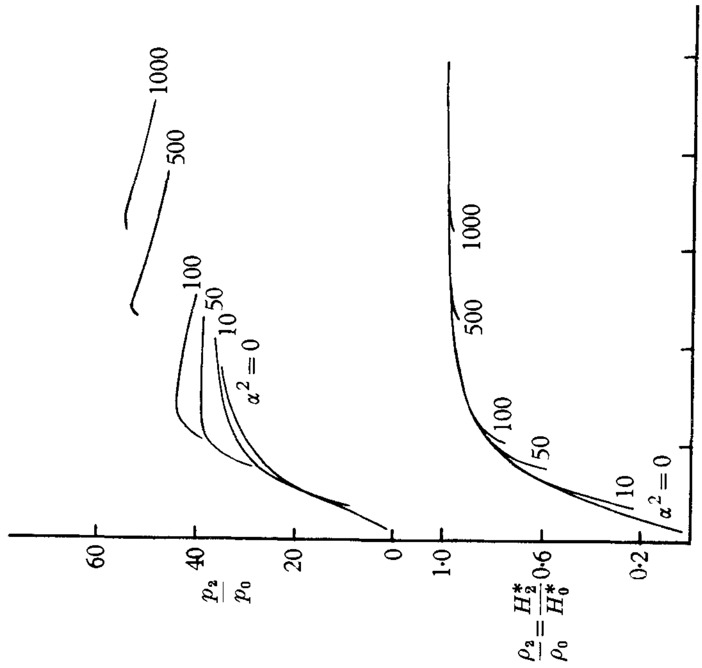

FIGURE 10. Burnt gas state ($Q/a_0^2 = 50$).

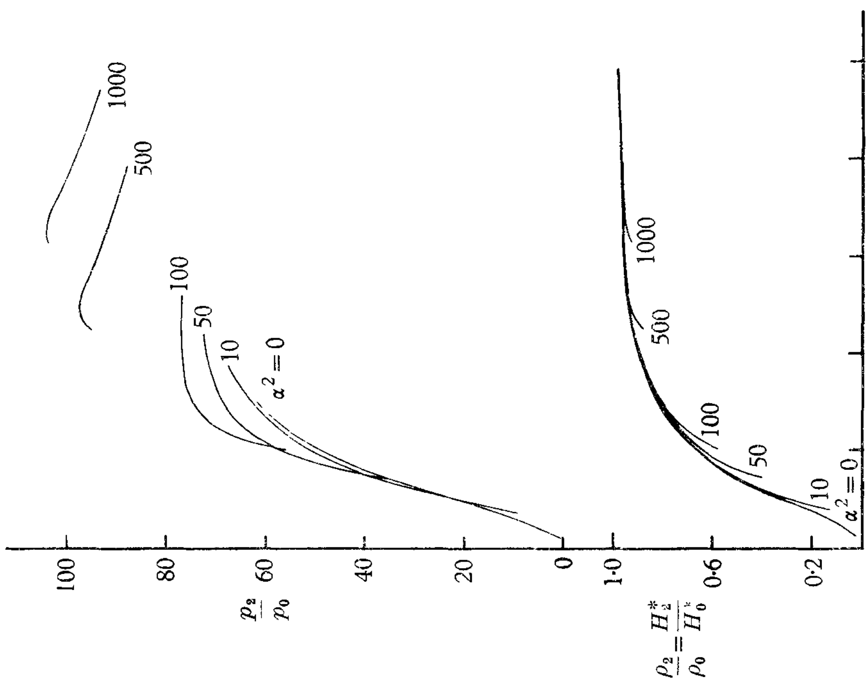

FIGURE 11. Burnt gas state ($Q/a_0^2 = 100$).

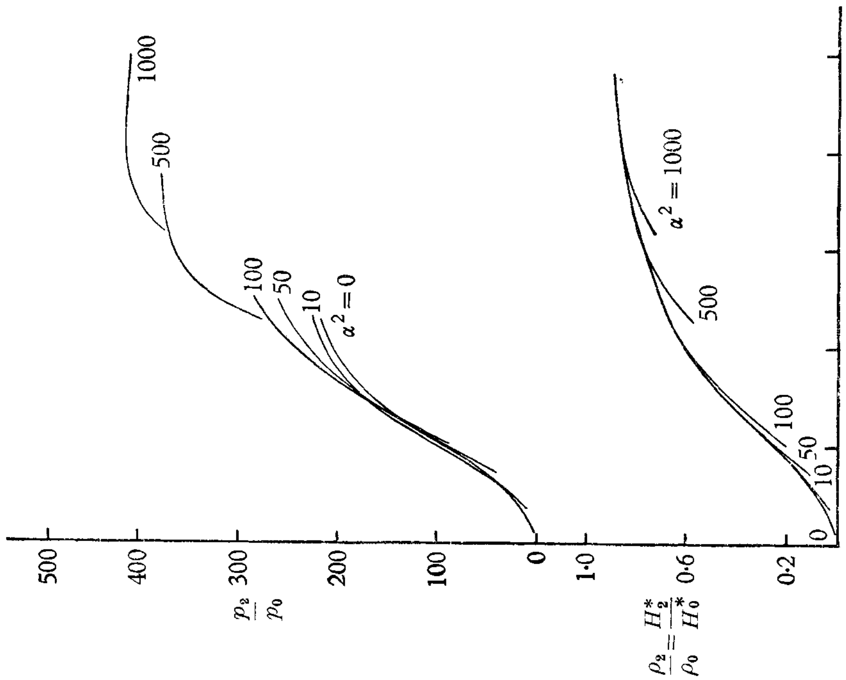

FIGURE 12. Burnt gas state ($Q/a_0^2 = 500$).

rapidly with increase of shock speed as H_0^* increases. As the exothermal energy release, Q/a_0^2, increases so does the range of possible wave speeds. In the products of combustion the 'sonic' speed is sensibly independent of the wave speeds. It increases with increase of H_0^* as is to be expected since it contains a contribution from the Alfvén speed. The magnitude of the increase is approximately independent of Q/a_0^2.

From figures 10, 11, 12 it can be seen that with large upstream magnetic fields the density and magnetic field in the products of combustion rapidly approach their asymptotic values as the shock speed increases, though more slowly with increase of exothermal energy release. The sole difference from an ordinary gasdynamic deflagration occurs in the pressure of the burnt gas. Whilst the pressure ratio p_2/p_0 is always greater than that in the ordinary gasdynamic case at shock speeds correspondingly greater than the minimum, for sufficiently large magnetic fields this ratio falls with increase of shock velocity (provided the exothermal energy is not too large) as a consequence of the interaction of the electromagnetic energy and stress with their mechanical counterparts.

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